Orange Juice Problem

Mathematical goals

This lesson unit is intended to help you assess how well students are able to reason proportionally in a realistic context: the recipe that will result in a stronger orange taste. In particular, this unit aims to identify and help students who have difficulties with:

- Choosing an appropriate, often informal, strategy to solve proportions.
- Seeing how quantities vary together and are able to see how the variation in one coincides with the variation in another.
- Recognizing ratios as distinct entities representing a relationship different from the quantities they compare.

Common Core State Standards

This lesson involves a range of *mathematical practices* from the standards, with emphasis on:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

This lesson asks students to select and apply mathematical content from across the grades, including the *content standards:*

6-RP: **Understand ratio concepts and use ratio reasoning to solve problems.**
7-RP: **Analyze proportional relationships and use them to solve real-world and mathematical problems.**
8-EE: **Understand the connections between proportional relationships, lines, and linear equations.**

This lesson meets the intent of the 7th grade content standards for the domain of Ratio and Proportional Relationships. However, 6th grade teachers could use this lesson to formatively assess students after teaching the standards in the 6th grade domain of Ratios and Proportional Relationships. According to Van de Walle, "many of the most valuable activities to develop proportional reasoning do not involve solving proportions at all but rather comparing ratios in similar but nonproportional settings." It would be suitable for 6th grade teachers to assess their students' proportional thinking at this point. 8th grade teachers might use this lesson to formatively assess students' proportional thinking at the beginning or during the teaching of the cluster 8-EE. This lesson could be used to connect proportional thinking to algebraic interpretations.

Introduction

This lesson unit is structured in the following way:

• Before the lesson, students attempt the task individually. You then review their work and formulate questions for students to answer in order for them to improve their work.
• At the start of the lesson, students work individually to answer your questions.

• Next, they work collaboratively, in small groups, to produce a better collective solution than those they produced individually. Throughout their work, they justify and explain their decisions to peers.
• In the same small groups, students critique examples of other students' work.
• In a whole-class discussion, students explain and compare the alternative approaches they have seen and used.
• Finally, students work alone again to improve their individual solutions.

Materials required

• Each individual student will need two copies of the worksheet *Orange Juice Problem*.
• Each small group of students will need a copy of *Sample Responses to Discuss* and the samples of student work provided.

Time needed

Approximately fifteen minutes before the lesson, a one-hour lesson, and ten minutes in a follow-up lesson (or for homework). **All timings are approximate. Exact timings will depend on the needs of the class.**

Before the lesson

Assessment task:

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Orange Juice Problem*.

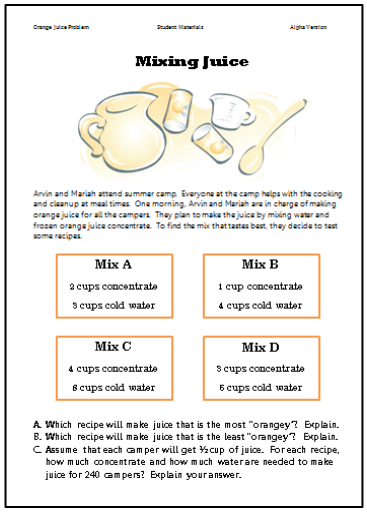
Introduce the task briefly and help the class to understand the problem and its context. You may want to show them a picture of frozen juice concentrate in case some have never made juice and may not know what concentrate is.

Think of a time when you mixed water and a powder or concentrate to make Koolaid, a sports drink, or juice.

Think about how the drink tasted if you put in too much water or not enough water.

Spend fifteen minutes on your own, answering these questions about juice that is made from water and concentrate.

Show your work. Don't worry if you can't figure it out. There will be a lesson on this material [tomorrow] that will help you improve your work. Your goal is to be able to answer this question with confidence by the end of that lesson.


Orange Juice Problem Student Materials Alpha Version

Mixing Juice

Arvin and Mariah attend summer camp. Everyone at the camp helps with the cooking and cleanup at meal times. One morning, Arvin and Mariah are in charge of making orange juice for all the campers. They plan to make the juice by mixing water and frozen orange juice concentrate. To find the mix that tastes best, they decide to test some recipes.

Mix A	Mix B
2 cups concentrate 3 cups cold water	1 cup concentrate 4 cups cold water
Mix C	**Mix D**
4 cups concentrate 8 cups cold water	3 cups concentrate 6 cups cold water

A. Which recipe will make juice that is the most "orangey"? Explain.
B. Which recipe will make juice that is the least "orangey"? Explain.
C. Assume that each camper will get ½ cup of juice. For each recipe, how much concentrate and how much water are needed to make juice for 240 campers? Explain your answer.

It is important that students answer the question without assistance, as far as possible. Students who sit together often produce similar answers, and then, when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of proportional reasoning and their different strategies. The purpose of this is to forewarn you of the issues that will arise during the lesson, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this is counterproductive, as it encourages students to compare scores and distracts their attention from how they may improve their mathematics.

Instead, help students to make further progress by asking questions that focus attention on aspects of their work. Some suggestions for these are given on the next page. These have been drawn from common misconceptions involving ratios and proportional reasoning.

We suggest that you write your own lists of questions, based on your own students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues - Suggested questions and prompts:

Common Issues	Suggested questions and prompts
Additive vs Multiplicative relationships	• ***If you've only thought about the amount of juice concentrate, how might the amount of water added to that concentrate affect the taste of the mixed juice?*** • ***What percentage of the total juice mix is the concentrate?***
No sense of covariation. They do not understand relationship in which two quantities vary together and cannot see how the variation in one coincides with the variation in another.	• ***Which recipe has a larger proportion of orange concentrate?*** • ***What is the unit rate (cups of water per cup of concentrate)?*** • ***Can you make fractions for each recipe comparing the concentrate to the water?*** • ***Can you make fractions for each recipe comparing the concentrate to the total amount of juice?*** • ***What percentage of the juice is concentrate?***
Does not understand different strategies for solving proportions or comparing ratios; can only solve one way.	• ***How did this group compare the amount of concentrate and the amount of water?***

Suggested lesson outline

Improve individual solutions to the assessment task (approximately 10 minutes)

Return your students' work on the *Orange Juice Problem*. Ask students to re-read both the *Orange Juice Problem* and their solutions. If you have not added questions to students' work, write a short list of your most common questions on the board. Students can then select a few questions appropriate to their own work and begin answering them.

> ***Recall what we were working on previously. What was the task?***

Draw students' attention to the questions you have written.

> ***I have read your solutions and I have some questions about your work.***
>
> ***I would like you to work on your own to answer my questions for ten minutes.***

Collaborative small-group work (approximately 10 minutes)
Organize the class into small groups of two or three students, and give out a fresh piece of paper to each group.

Ask students to try the task again, this time combining their ideas.

> ***Put your own work aside until later in the lesson. I want you to work in groups now.***
>
> ***Your task is to produce a solution that is better than your individual solutions.***

While students work in small groups **you have two tasks**, to note different student approaches to the task, and to support student problem solving. You might want to use a monitoring tool to note the different strategies and the order you would like to ask them to present their work.

Note different student approaches to the task
Notice how students decide which recipe will be stronger and weaker tasting. Notice what strategies they use for use in whole-class discussion. You can use this information to focus the whole-class plenary discussion towards the end of the lesson.

In particular, note any common mistakes. For example, are students consistently focusing on the difference between the two numbers, the additive relationship?

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, you could write a relevant question on the board. You might also ask a student who has performed well on one part of the task to help a student struggling with that part of the task.

The following questions and prompts would be helpful:

> ***What information have you been given?***
>
> ***What do you need to find out?***
>
> ***If you've only thought about the amount of juice concentrate, how might the amount of water added to that concentrate affect the taste of the mixed juice?***

How would the taste of the juice change if you added one more cup of water to this recipe? If you added one more cup of concentrate instead?

How will you compare the tastes of each of the four recipes?

Why do you think your conjecture might be true?

Students might argue that the recipes with a greater difference of concentrate and water will taste more "orangey," thus indicating they are only focusing on the additive relationship. If this issue arises in your class, help the student to focus his or her attention on the multiplicative relationship by asking:

Can you think of a different way of looking at the recipes?

If the student(s) cannot offer another way, ask

Which recipe has a larger proportion of orange concentrate?

What percentage of the total juice mix is the concentrate?

If students have found the correct answers to **A** and **B**, focus their attention on exploring alternative methods. Check to see if students who have one correct method can still see the validity of another. Without this ability to be flexible with different arguments, students have not reached a desired level of proportional reasoning.

What is the unit rate (cups of water per cup of concentrate)?

Can you make fractions for each recipe comparing the concentrate to the water?

Can you make fractions for each recipe comparing the concentrate to the total amount of juice?

What percentage of the juice is concentrate?

Make sure that students can both create and justify their solution(s). Some stronger explanations are shown in the *Sample Responses to Discuss*. Do **NOT** give the students the correct answer yet.

Make a note of student approaches to the task

Give each small group of students a copy of the *Sample Responses to Discuss*. Choose the samples of student work that match your students' level of understanding. Display the following questions on the board or OHP using the provided sheet: *Analyzing Student responses to discuss*.

Describe the strategy the student used.

You might, for example:

• Describe the way the student has compared the concentrate and water.

• Describe what the student did to calculate which juice was strongest.

• Explain what the student could do to make his or her solution correct or clearer if they calculated correctly.

This analysis task will give students an opportunity to evaluate a variety of alternative approaches to the task, without providing a complete solution strategy.

During small-group work, support student thinking as before. Also, check to see which of the explanations students find more difficult to understand. Identify one or two of these approaches to

discuss in the plenary discussion. Note similarities and differences between the sample approaches and those the students took in small-group work.

Plenary whole-class discussion comparing different approaches (approximately 20 minutes)

Organize a whole-class discussion to consider different approaches to the task. The intention is for you to focus on getting students to understand the methods for finding answers, rather than numerical solutions. **Focus your discussion on parts of the two small-group tasks students found difficult.**

> ***Let's stop and talk about different approaches.***

Ask the students to compare the different solution methods. Still do NOT give them the correct answer during the discussion.

> ***Which approach did you like best? Why?***
>
> ***Which approach did you find it most difficult to understand?***
>
> ***Sami, your group used that method. Can you explain that for us?***

Improving individual solutions to the assessment task (10 minutes)

If you are running out of time, you could schedule this activity for the next lesson or for homework.

Make sure students have their original individual work on the *Orange Juice Problem* task on hand. Give them a fresh, blank copy of the *Orange Juice Problem* answer sheet.
If a student is satisfied with his or her solution, ask the student to try a different approach to the problem and to compare the approach already used.

> ***Read through your original responses and think about what you have learned this lesson. Using what you have learned, try to improve your work.***

This Formative Assessment Lesson was created around tasks taken from *Connected Mathematics* and *Elementary and Middle School Mathematics – Teaching Developmentally*.

Solution

Most orangey mix is **Mix A**. Least orangey mix is **Mix B**.

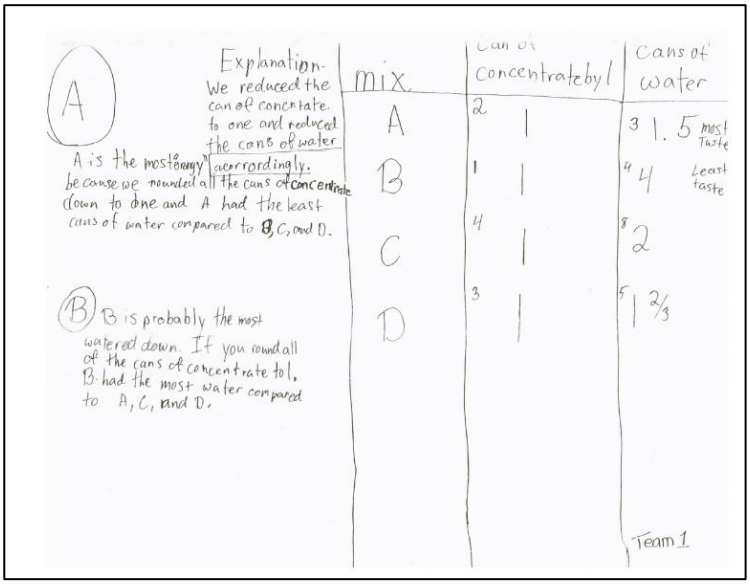
Teams 1, 3, 5, and 7 are correct. Any of these strategies would be suitable for solving the Orange Juice Problem.

Analysis of *Student Responses to Discuss*

Team 1's Method

Team 1 created a table to organize their data. Their strategy was to calculate unit rates (cans of water per can of concentrate) and compare those.

Team 1's Solution



A

Explanation- We reduced the can of concentrate to one and reduced the cans of water accordingly.

A is the most "orangy" because we rounded all the cans of concentrate down to one and A had the least cans of water compared to B, C, and D.

B

B is probably the most watered down. If you round all of the cans of concentrate to 1, B had the most water compared to A, C, and D.

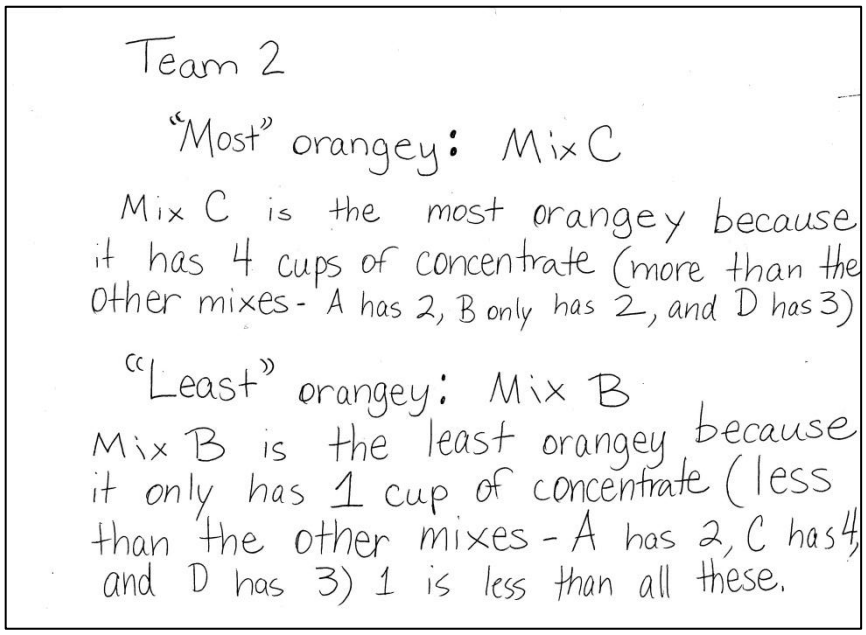
mix	Can of concentrate by 1	Cans of water
A	2 | 1	3 1.5 most Taste
B	1 | 1	4 4 Least taste
C	4 | 1	8 2
D	3 | 1	5 1 2/3

Team 1

Team 2's Method

Team 2 only considered the juice that had the most orange juice concentrate. They didn't think about how the cups of water would affect the taste. They are not comparing the two parts thus not considering all constraints of the problem.

Team 2's Solution



Team 2

"Most" orangey: Mix C

Mix C is the most orangey because it has 4 cups of concentrate (more than the other mixes - A has 2, B only has 2, and D has 3)

"Least" orangey: Mix B

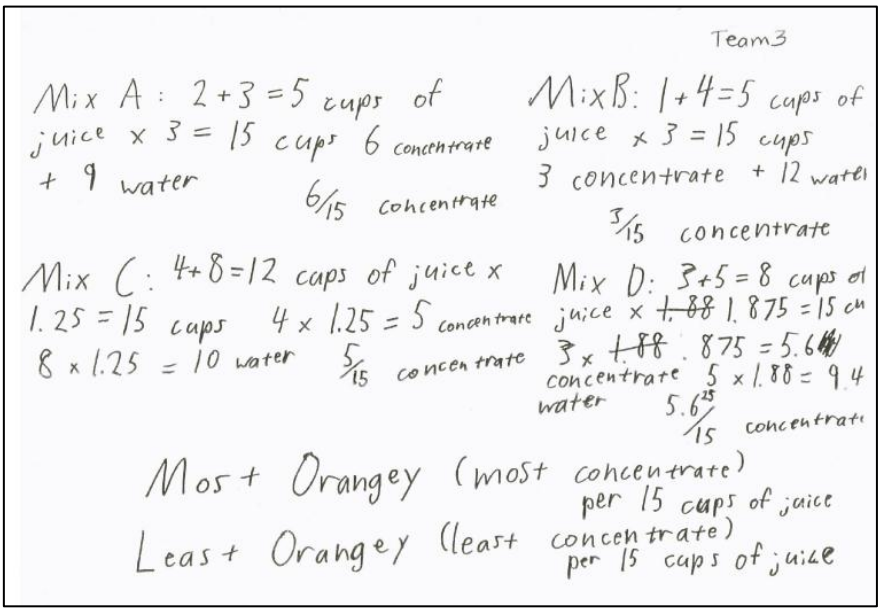
Mix B is the least orangey because it only has 1 cup of concentrate (less than the other mixes - A has 2, C has 4, and D has 3) 1 is less than all these.

Team 3's Method

Team 3 compared the four juices by using

15 total cups of juice for each recipe. They used proportional reasoning to calculate the number of cups of water and juice concentrate for each recipe.

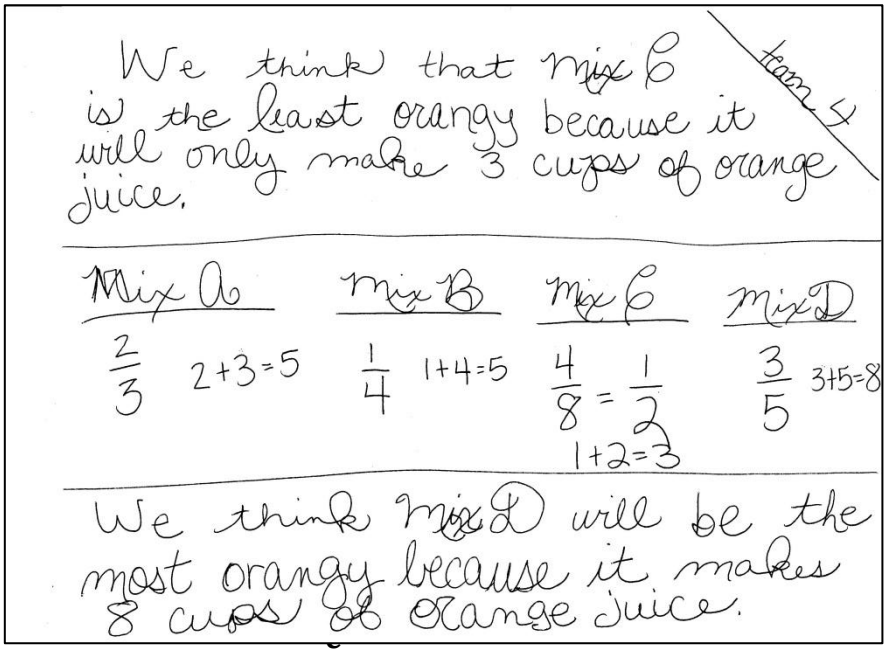
Team 3's Solution



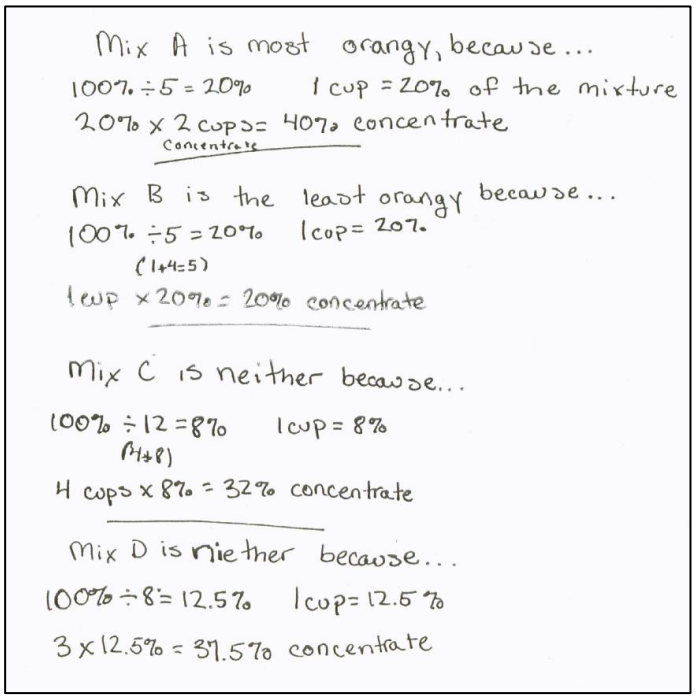
Team 4's Method

Team 4 calculates the total amount of juice without considering how the two amounts of ingredients compare to change the taste of the juice.

Team 4's Solution



Team 5's Solution



Mix A is most orangy, because...

100% ÷ 5 = 20% 1 cup = 20% of the mixture

20% x 2 cups concentrate = 40% concentrate

Mix B is the least orangy because...

100% ÷ 5 = 20% 1 cup = 20%

(1+4=5)

1 cup x 20% = 20% concentrate

Mix C is neither because...

100% ÷ 12 = 8% 1 cup = 8%

(4+8)

4 cups x 8% = 32% concentrate

Mix D is niether because...

100% ÷ 8 = 12.5% 1 cup = 12.5%

3 x 12.5% = 37.5% concentrate

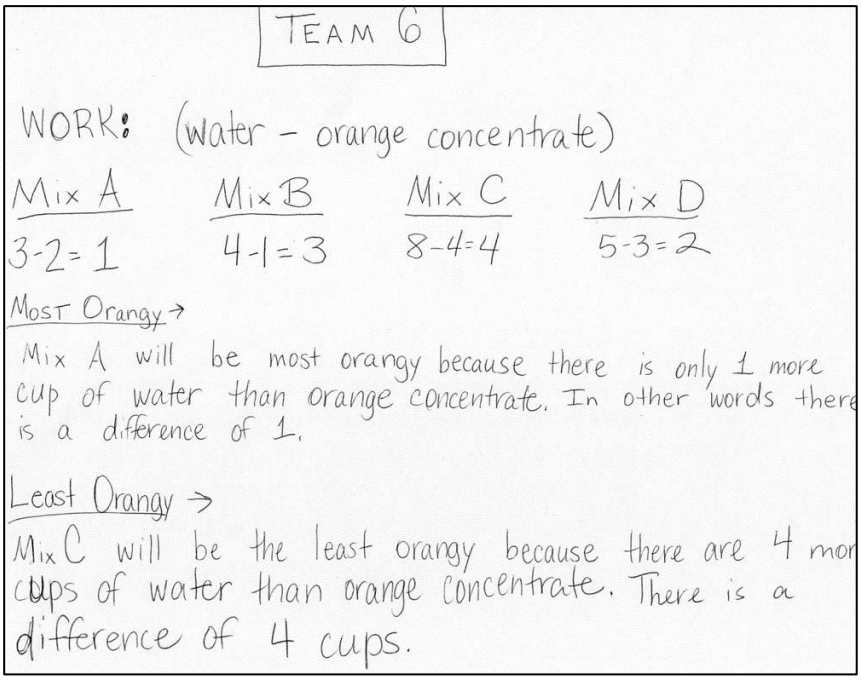
Team 5's Method

Team 5 correctly calculated the percentage of each Mix. They knew that the total amount of juice, 100%, could be divided by the total cups made to figure the percentage of 1 cup of mixed juice. This group then multiplied that answer by the # cups of juice concentrate to calculate how much of the total mix was concentrate.

Team 6's Method

Team 6 is thinking about the additive relationship instead of the multiplicative relationship between the water and the juice concentrate. They subtracted the amounts of each instead of thinking of them as ratios using proportional reasoning.

Team 6's Solution



TEAM 6

WORK: (water - orange concentrate)

Mix A	Mix B	Mix C	Mix D
3-2=1	4-1=3	8-4=4	5-3=2

Most Orangy →

Mix A will be most orangy because there is only 1 more cup of water than orange concentrate. In other words there is a difference of 1.

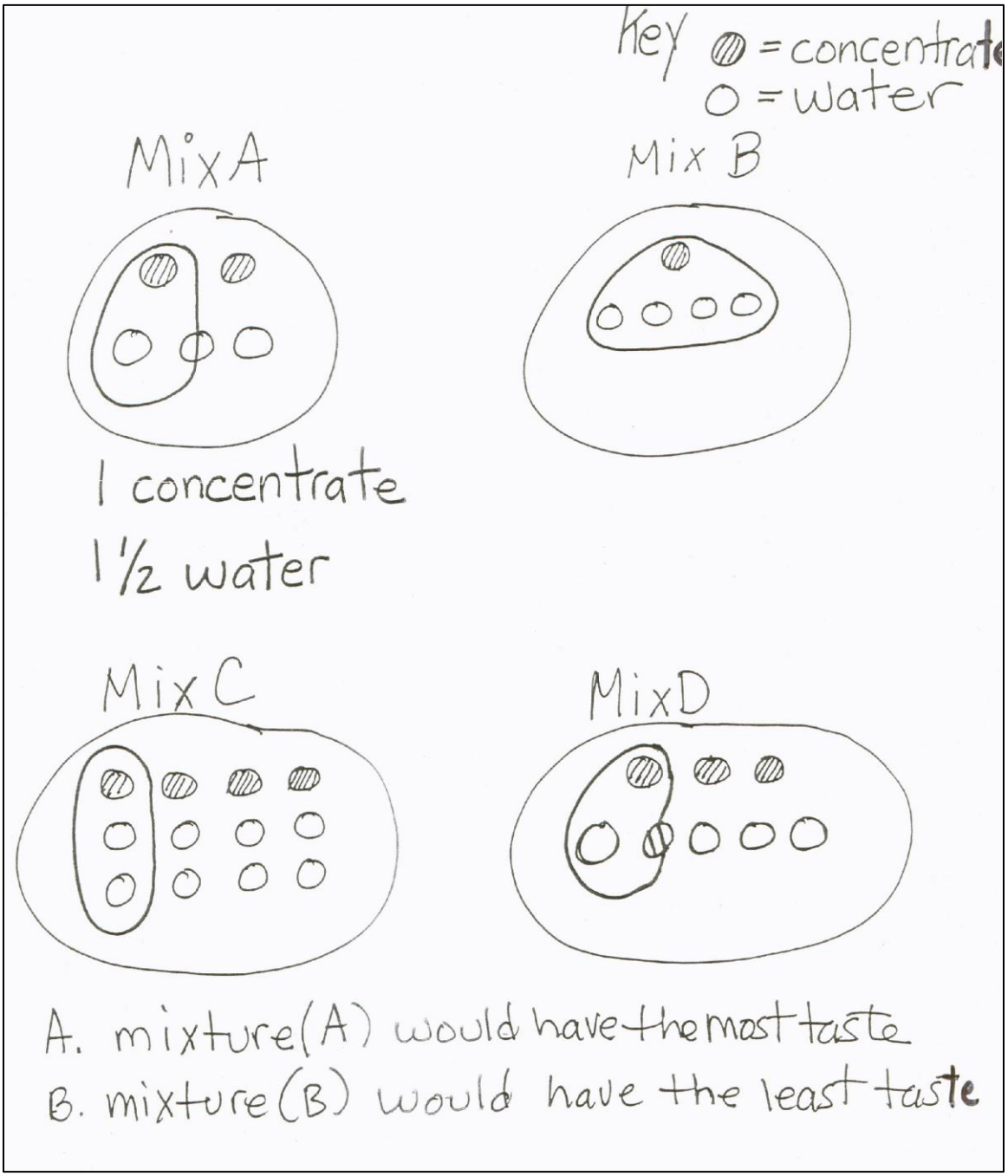
Least Orangy →

Mix C will be the least orangy because there are 4 mor cups of water than orange concentrate. There is a difference of 4 cups.

Team 7's Method

Team 7 used the strategy "Draw a Picture" to solve this problem.

Team 7's Solution



Mixing Juice

Arvin and Mariah attend summer camp. Everyone at the camp helps with the cooking and cleanup at meal times. One morning, Arvin and Mariah are in charge of making orange juice for all the campers. They plan to make the juice by mixing water and frozen orange juice concentrate. To find the mix that tastes best, they decide to test some recipes.

Mix A	Mix B
2 cups concentrate	1 cup concentrate
3 cups cold water	4 cups cold water

Mix C	Mix D
4 cups concentrate	3 cups concentrate
8 cups cold water	5 cups cold water

Answer the following on a separate sheet of paper.

A. Which recipe will make juice that is the most "orangey"? Explain.

B. Which recipe will make juice that is the least "orangey"? Explain.

Sample Responses to Discuss

Here is some work on the *Orange Juice Problem* from students in another class.

For each piece of work:

1. Write the name of the student whose solution you are analyzing.
2. Describe the problem solving approach the student used.

   For example, you might:

   - Describe the way the student has compared the concentrate and water.
   - Describe what the student did to calculate which juice was strongest.

3. Explain what the student needs to do to complete or correct his or her solution.

_____’s_ Solution

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_____’s_ Solution

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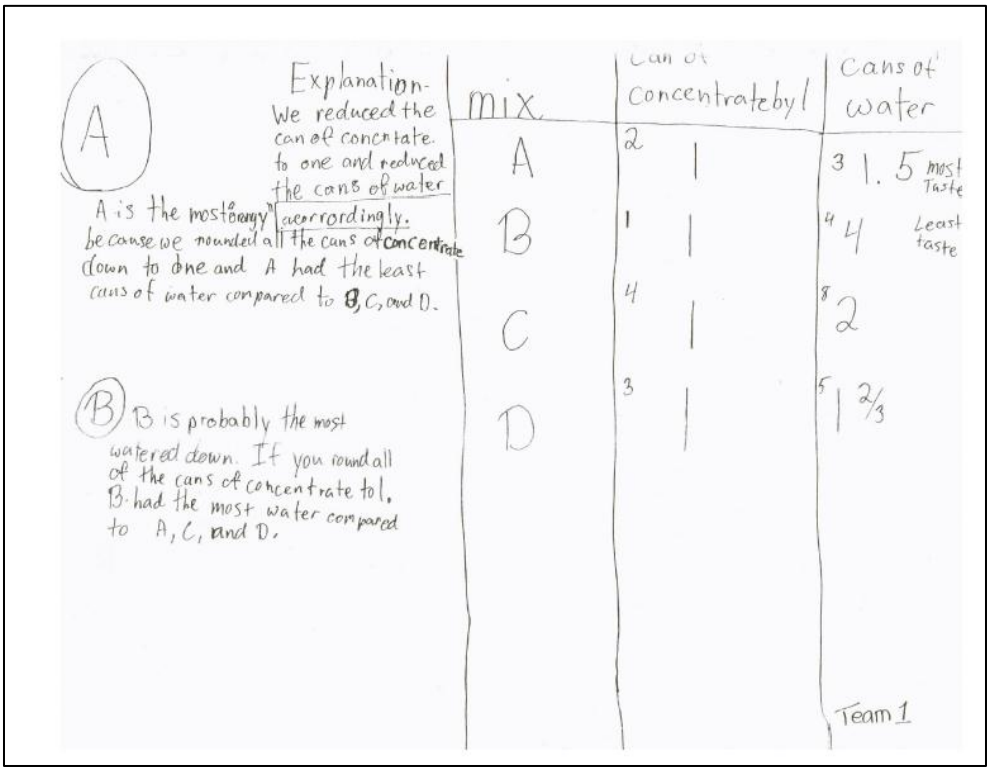
Team 1's Solution



A

Explanation- We reduced the can of concentate to one and reduced the cans of water accordingly.

A is the most "orangy" because we rounded all the cans of concentrate down to one and A had the least cans of water compared to B, C, and D.

B

B is probably the most watered down. If you round all of the cans of concentrate to 1, B had the most water compared to A, C, and D.

mix	Can of concentrate by 1	Cans of water
A	2 1	3 1.5 most Taste
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Team 1

Team 2's Solution

Team 2

"Most" orangey: Mix C

Mix C is the most orangey because it has 4 cups of concentrate (more than the other mixes - A has 2, B only has 2, and D has 3)

"Least" orangey: Mix B

Mix B is the least orangey because it only has 1 cup of concentrate (less than the other mixes - A has 2, C has 4, and D has 3) 1 is less than all these.

Team 3's Solution

Team3

Mix A: 2+3=5 cups of juice x 3 = 15 cups 6 concentrate + 9 water

6/15 concentrate

Mix B: 1+4=5 cups of juice x 3 = 15 cups 3 concentrate + 12 water

3/15 concentrate

Mix C: 4+8=12 cups of juice x 1.25 = 15 cups 4 x 1.25 = 5 concentrate 8 x 1.25 = 10 water

5/15 concentrate

Mix D: 3+5=8 cups of juice x ~~1.88~~ 1.875 = 15 cu 3 x ~~1.88~~ .875 = 5.6 concentrate 5 x 1.88 = 9.4 water

5.6 25/15 concentrate

Most Orangey (most concentrate per 15 cups of juice

Least Orangey (least concentrate per 15 cups of juice

Team 4's Solution

Team 4

We think that mix C is the least orangy because it will only make 3 cups of orange juice.

Mix A	Mix B	Mix C	Mix D
$\frac{2}{3}$ 2+3=5	$\frac{1}{4}$ 1+4=5	$\frac{4}{8} = \frac{1}{2}$ 1+2=3	$\frac{3}{5}$ 3+5=8

We think mix D will be the most orangy because it makes 8 cups of orange juice.

Team 5's Solution

Mix A is most orangy, because...

100% ÷ 5 = 20% 1 cup = 20% of the mixture

20% x 2 cups = 40% concentrate
Concentrate

Mix B is the least orangy because...

100% ÷ 5 = 20% 1 cup = 20%
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Team 6's Solution

TEAM 6

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Team 7's Solution